Clever Stuff For Common Problems

Going beyond simple algorithms

Toy Problems For The Real World

Classroom Resources

BY NC SA

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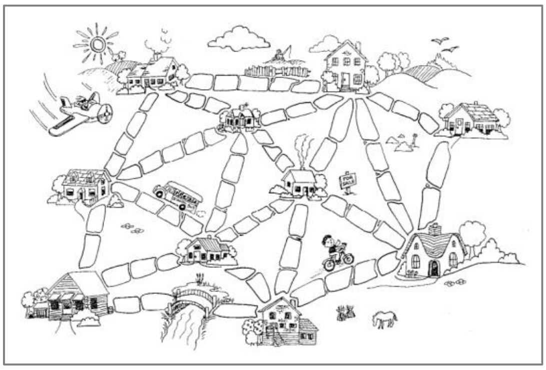
This simple class activity leads to students discovering either Prim's or Kruskal's algorithm for finding a Minimum Spanning Tree (MST). It goes on to consider the similarities and differences between a MST and other classic graph traversal algorithms.

Preparation required:
Muddy City puzzle sheet for each small group

A Muddy City

This simple exploration allow pupils to discover famous algorithms that are used today in a wide range of real world problems. Split into small groups and distribute the map. Explain that it has no roads, making it difficult (and muddy) to get around. Enough streets must be paved so it is possible to travel from any house to any other house, possibly via other houses. The paving should be done at a minimum cost. The diagram indicates the number of paving stones needed to connect each house. The bridge requires a paving slab. What is the smallest number of paving stones needed?



Give groups 10 minutes or so to come up with answers. 23 slabs are needed. Ask groups to indicate their solution on a projected map. There are several correct solutions. Two are shown as exemplars, but the key task is to get groups to articulate their strategy for solving it. Allow time for students to articulate and discuss alternatives. There are many possibilities but intuitively, students will probably come up with selecting the shortest connection first as a starting point. Two possible approaches generally follow, both known as 'greedy' algorithms – they always take the best option available. Each results in an optimal solution, known as a Minimum Spanning Tree (MST).

Prim's Algorithm

Prims algorithm selects the lowest connector from the two houses first connected. It is demonstrated in the subsequent slides, which can be introduced as a series of questions, each click building up the MST. It is worth becoming familiar with the slide notes which contain the narrative. Although known as Prims algorithm it was developed initially by a Czech mathematician, Vojtěch Jarník in 1930. Only later was it also discovered independently by computer scientists Robert Prim (1957) and Edsger Dijkstra (1959).

Kruskal's Algorithm

Prim's algorithm is not the only one for finding a Minimum Spanning Tree. Another, often discovered intuitively by children was developed by American computer scientist Joseph Kruskal in 1956. Challenge pupils to spot the difference between this approach and Prim's. A simple walkthrough is provided. As with Prim's we select the shortest edge as a starting point, but Kruskals algorithm takes the shortest connection, regardless of whether it joins the existing path.

Minimum Spanning Trees provide the shortest connection utilising existing points. Either algorithm will result in an optimal solution. They are widely used in many real world scenarios, a good example might be planning the network cabling in the school. Knowing they have discovered a 'real' algorithm can be very motivating for pupils. Can they think of other situations where finding a MST would be useful?

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Similar Problems

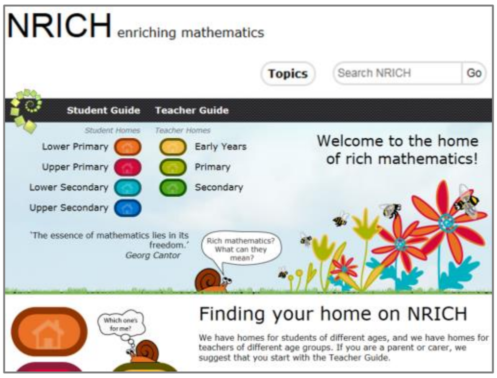
Many problems share similarities with a MST. A Steiner Tree shortens distances by introducing new interim points but an optimal solution is difficult to compute. A CS Unplugged outdoor activity, Ice Roads could make an extension activity. An explanation by NumberPhile: www.youtube.com/watch?v=dAyDi1aa40E is worth watching. Soap bubbles will naturally form a local Steiner Tree. It is simple to demonstrate with a little preparation.

The Create Maths 'Working For Efficiency' resources use real world graph problems. 'Cable Connections' involves finding a MST. Problems often appear similar, but can be subtly different. In Muddy City, the MST challenge finds the shortest tree that connected all nodes. 'Deliveries' looks for the shortest cycle - famously known as the Travelling Salesman Problem. It is famous because no known algorithm can solve it in a reasonable amount of time, even for a small number of locations. It is a common misconception that as computers get faster and faster, they will eventually be able to solve any problem such as this by 'brute force'. But even with 8 locations, all connected, a fast computer might take around 5 minutes to check every possible combination. Just increasing that to 10 requires 5 hours, and with 20 locations, it takes nearly 200 million years!

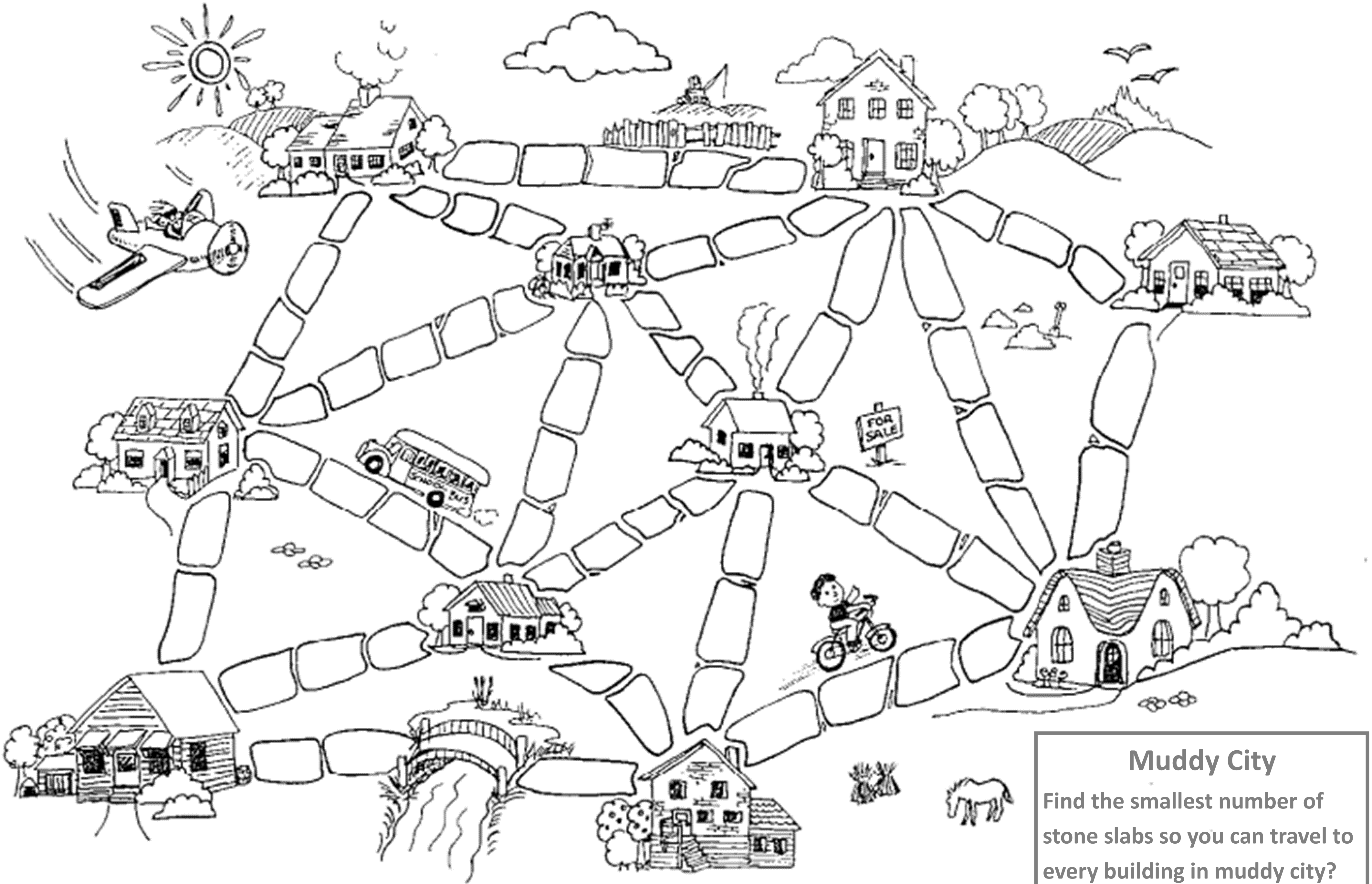
If this seems ridiculous, read the New Zealand CS Field Guide: www.csfieldguide.org.nz/en/chapters/complexity-tractability.html. It explains the concept of intractability well and has a nice interactive to show how such problems quickly becomes intractable. Nonetheless, having just discovered Prims algorithm, there is no harm in children having a go. With small graphs it is possible to get 'good' solutions, even if we can't prove they are optimal. Again, the focus is on developing a greedy strategy. The website www.math.uwaterloo.ca/tsp/games/index.html has two games for children to explore. Well worth investigating.

Is a task such as planning a paper round (the 3rd Create Maths activity) like any of the previous problems? Can we use a previous strategy to solve this? Sadly we can't. In this case, we need a solution that visits every edge once, rather than every node, and returns to the starting point. These tours are known as Eulerian Circuits, named after the mathematician Leonhard Euler (1707 -1783) and his first statement of this classic problem, the Bridges of Konigsberg. Edge traceable graphs make great 'pencil puzzles'. Students try to trace a route along all the lines without taking their pencil off the paper.

Like Hamiltonian Cycles, finding an Eulierian Circuit is a very difficult problem for a computer. But being able to say whether a graph contains an edge traceable circuit is much easier. It involves an investigation of the properties of the graph. Encourage children to study the number of connections at each node. Can they spot the pattern? The Create Maths teacher notes provide further pointers for structuring this investigation. Investigations of this sort are ideal for developing the ability to generalise: observing specific examples, spotting similarities and developing a general hypothesis that can be applied to all similar problems.



These algorithms fall under the heading of Discrete (or Decision) Maths and many lie at the heart of Computer Science. The Nrich Maths project (http://nrich.maths.org) also has many activities that can introduce Computational Thinking in KS3. The entire repository is searchable by Topic. A good starting point is to select Topics – Decision Mathematics – Network / Graph Theory. The results can then be filtered by Key Stage. Each resource has hints to get children going, solutions and teachers resources. The teacher notes, in particular, are very useful, providing links to extensions and clear explanations as to the purpose of the exercise.



Muddy City

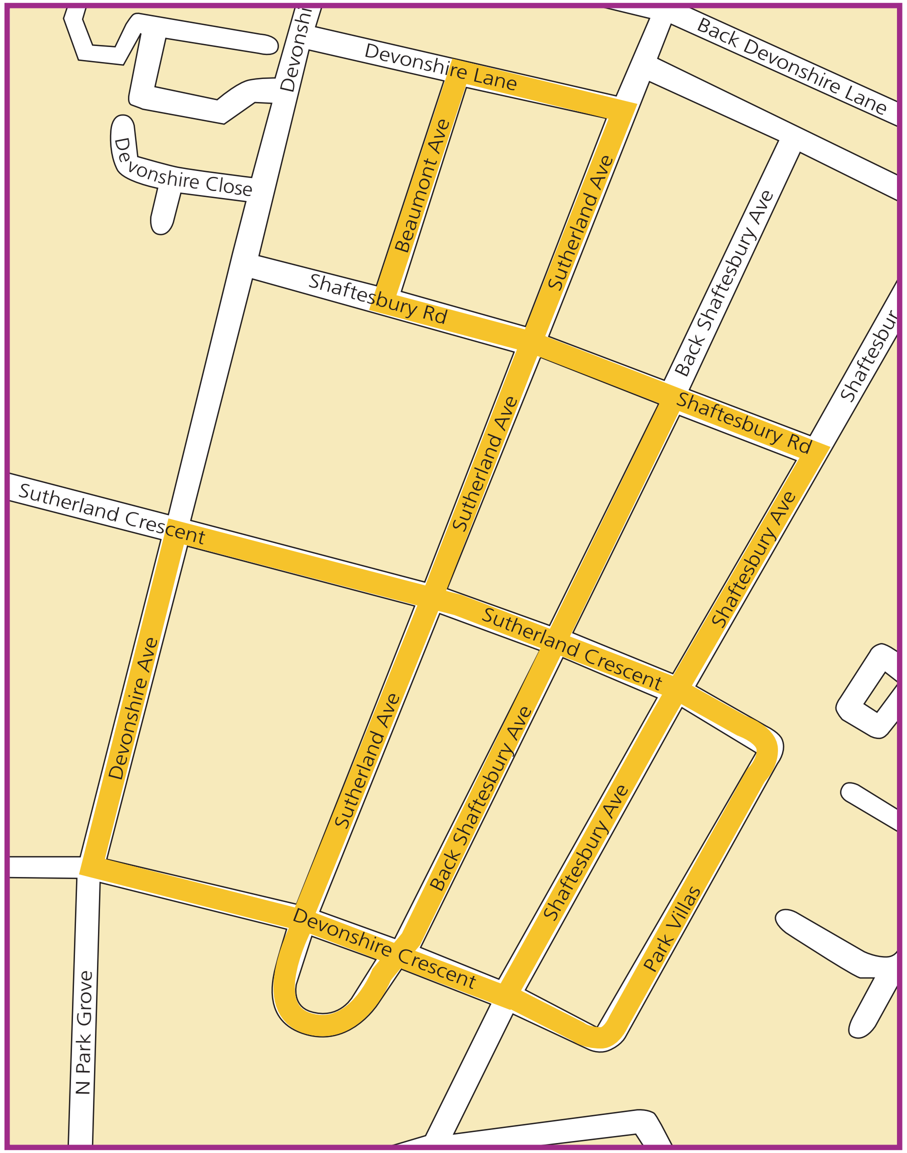
Find the smallest number of stone slabs so you can travel to every building in muddy city?

paper rounds

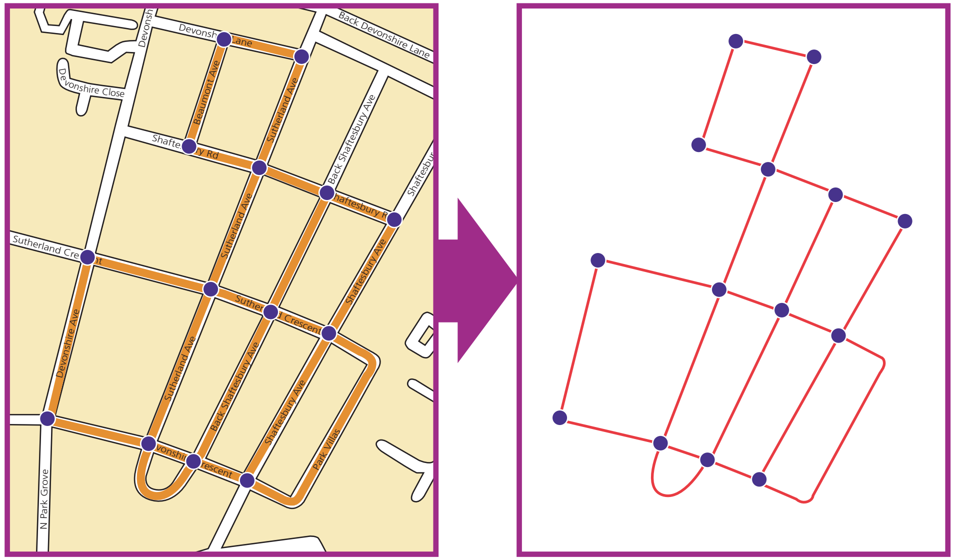
The **mathematics** here is often used at work. It solves a variety of networking problems.

Getting there

Can you find the best route to deliver to every street?



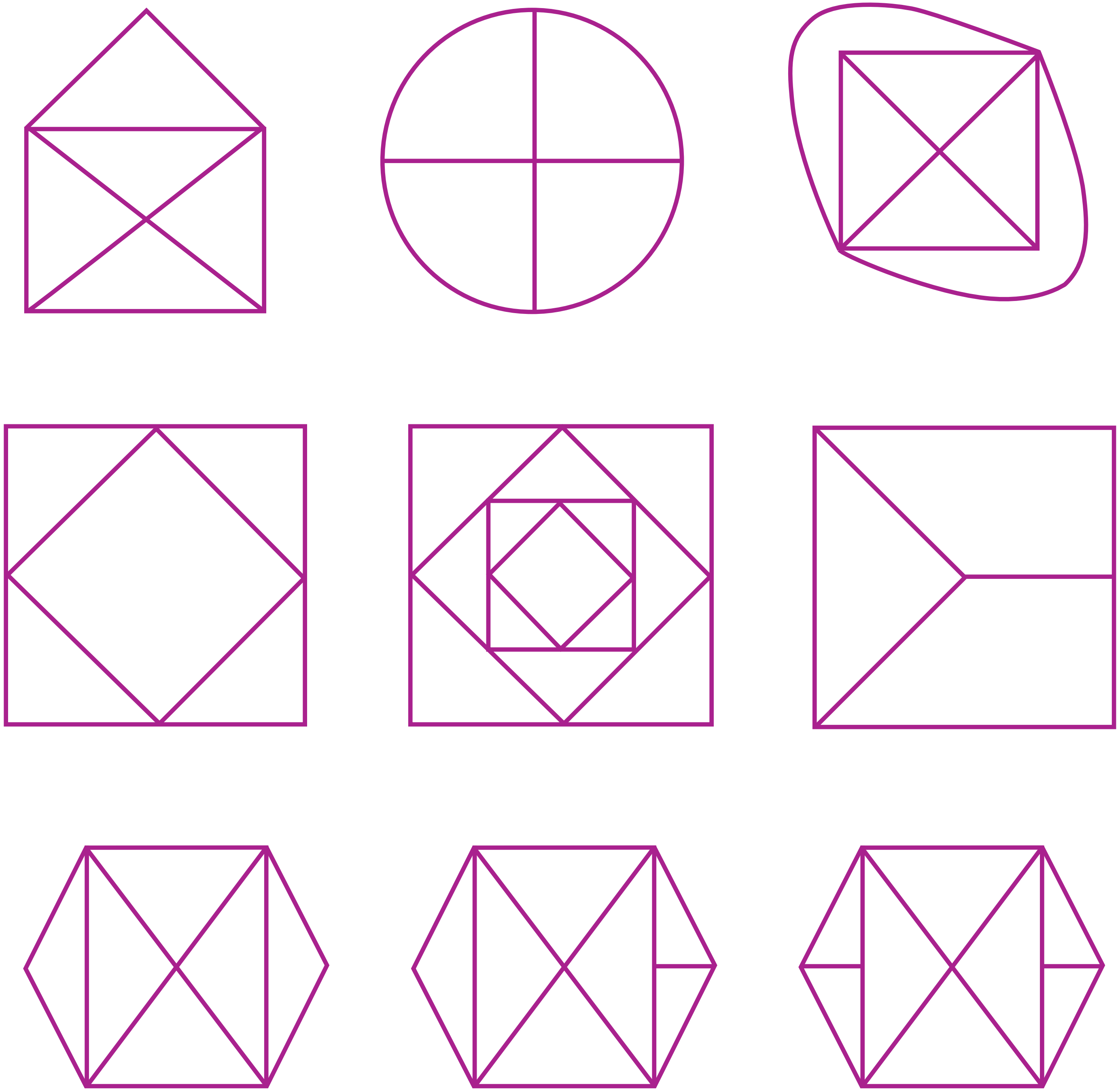
- The roads marked show Pat's paper round.
- The round is some way from Pat's home so the newsagent drops Pat off in her van.
- She also collects her at the end of her delivery.



- Pat can save time if she can find a route where she only walks along each road once.
- Where should Pat be dropped off by the newsagent and where should she be picked up at the end of her round?

Five of these networks can be traced without taking your pencil off the page.

Which ones?



Find a rule which predicts which networks are traceable.

Test your rule with some networks of your own.

Getting there

cable connections

The **mathematics** here is often used at work.
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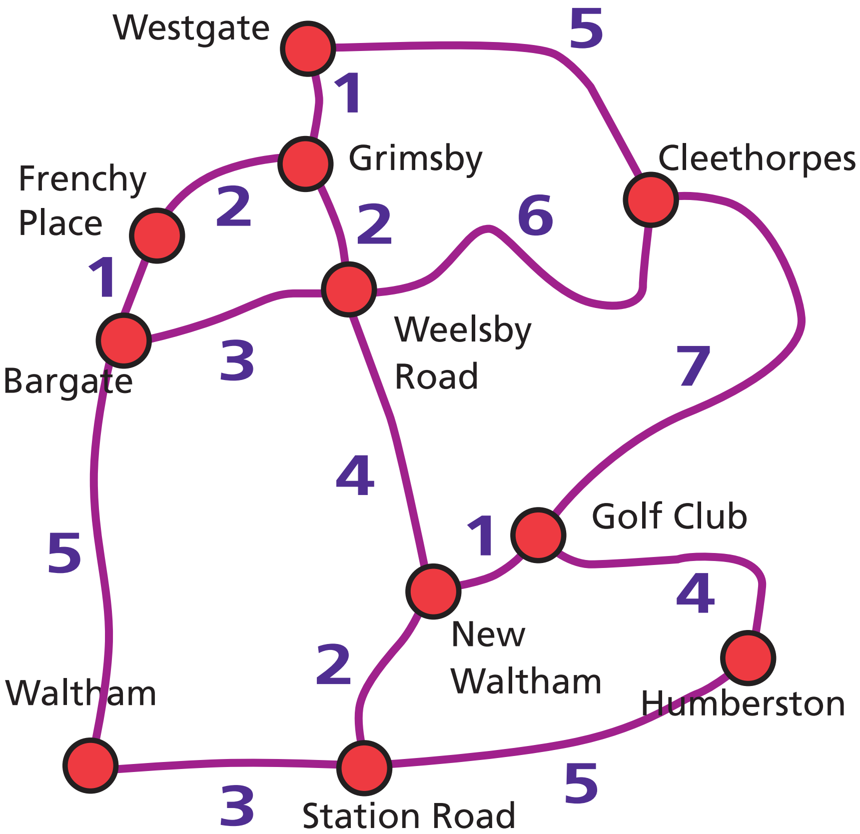
create maths

Getting there

Can you link these places using the shortest length of cable?

- **Your task** is to plan how to lay cables for a cable TV service between all the places on the map.
- **Laying cable** is very expensive so you need to keep the total length of cable as short as possible.

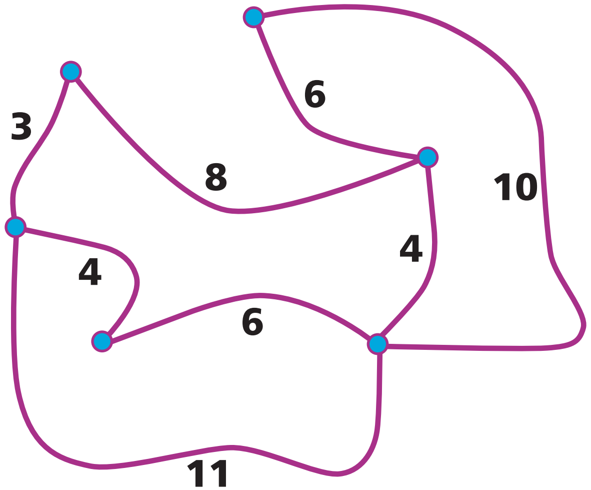


Distances in kilometres

- Make sure all the locations are **connected**. They do not have to be connected directly, as the signals travel along the cables at the **speed of light**!
- You discover that due to planned engineering works it will not be possible to lay cable directly between Weelsby Road and New Waltham.
- What is the **shortest length of cable** required now?

Find the minimum length of cable required to connect all the locations in each of these networks:

1



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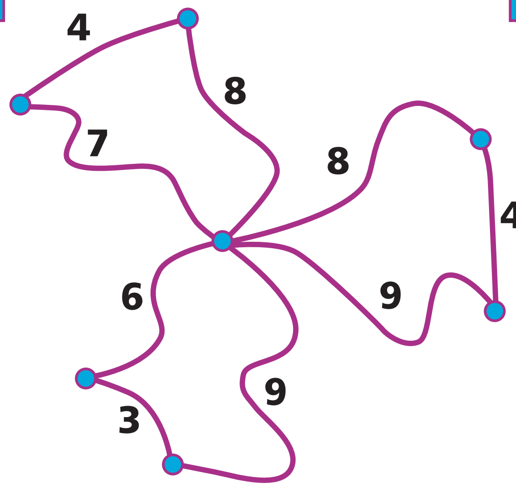

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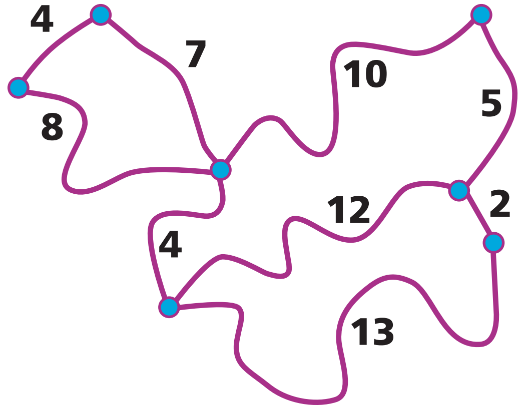

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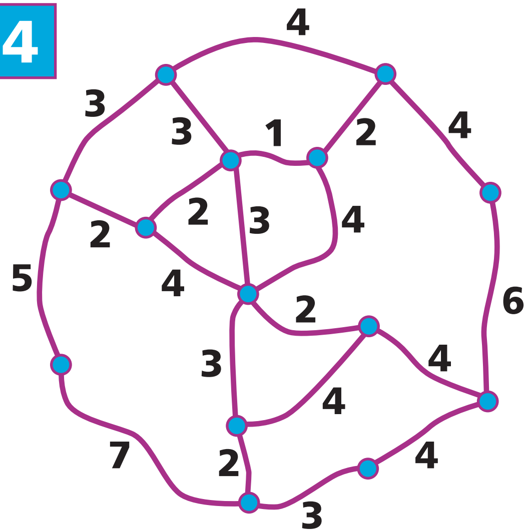

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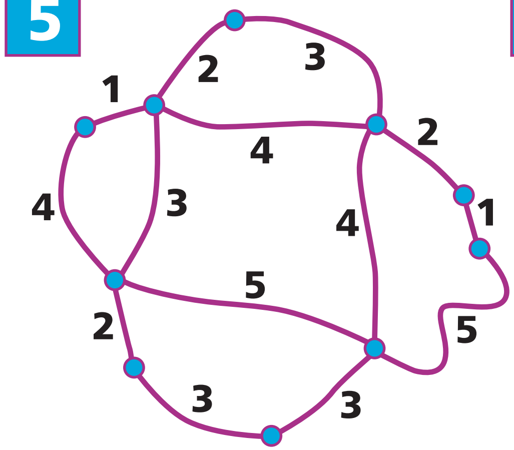

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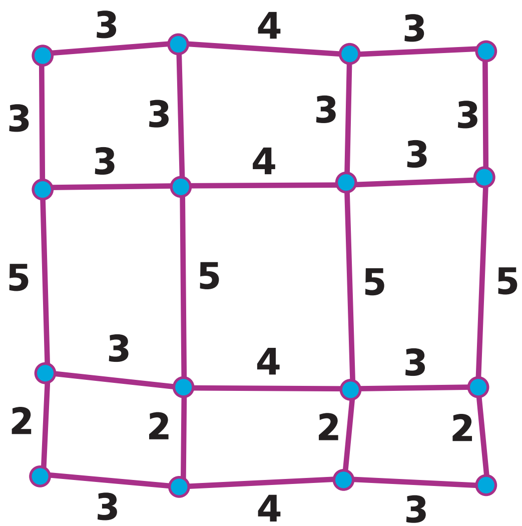

Getting there

deliveries

The **mathematics** here is often used at work.
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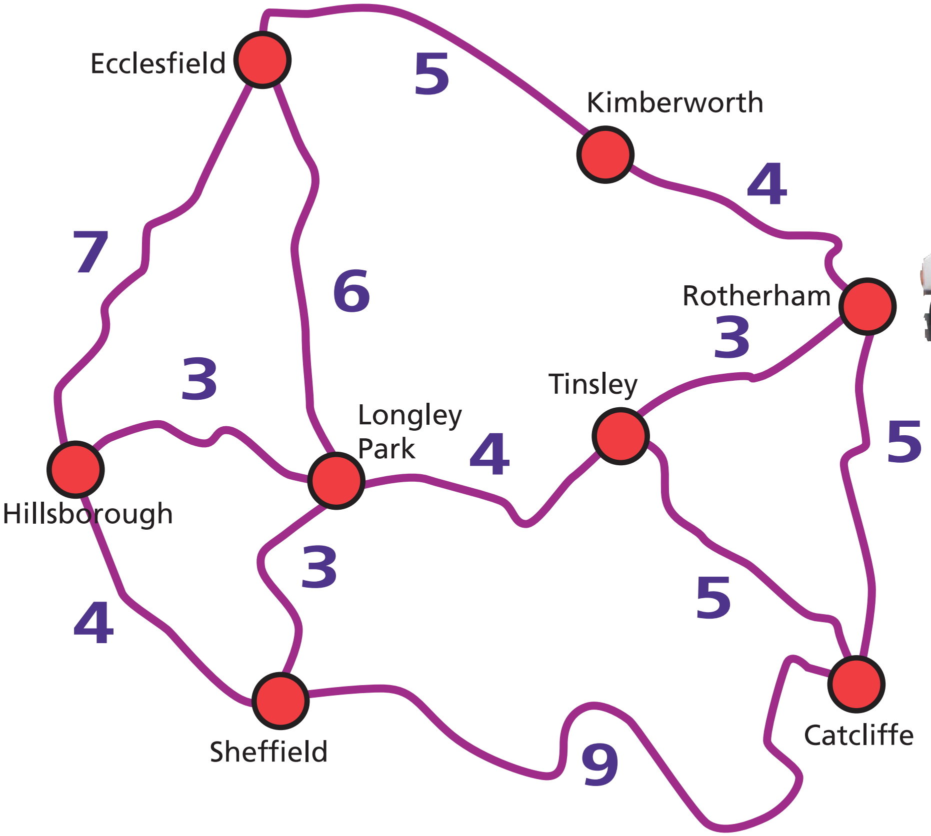
create maths

Getting there

Can you find the best route to deliver to every town?

- You have to make deliveries to all the places marked in the map, starting from your company's warehouse in Sheffield.



Distances in kilometres

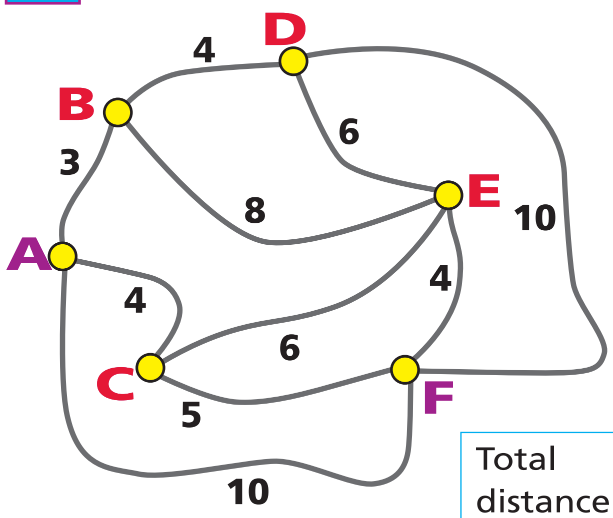
- The numbers beside the roads tell you the distances between the places.
- Your task is to find the shortest route to all the drop-off points, ending up back at the warehouse in Sheffield.

In each case, find the shortest route which visits every place.

Getting there

1

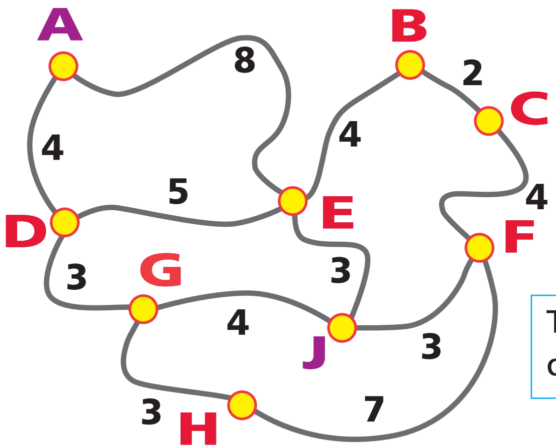
Start at **A** and end at **F**



Total distance = _____

2

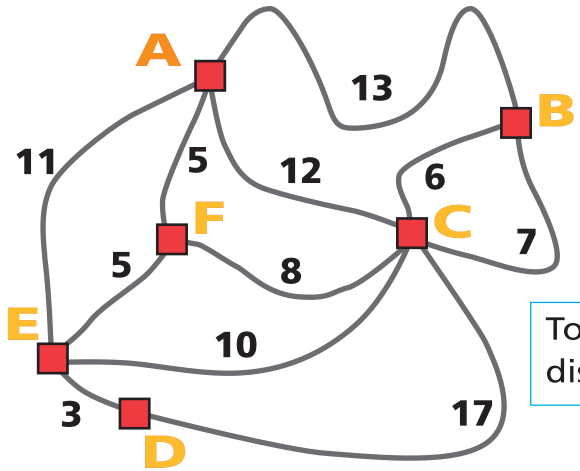
Start at **A** and end at **J**



Total distance = _____

3

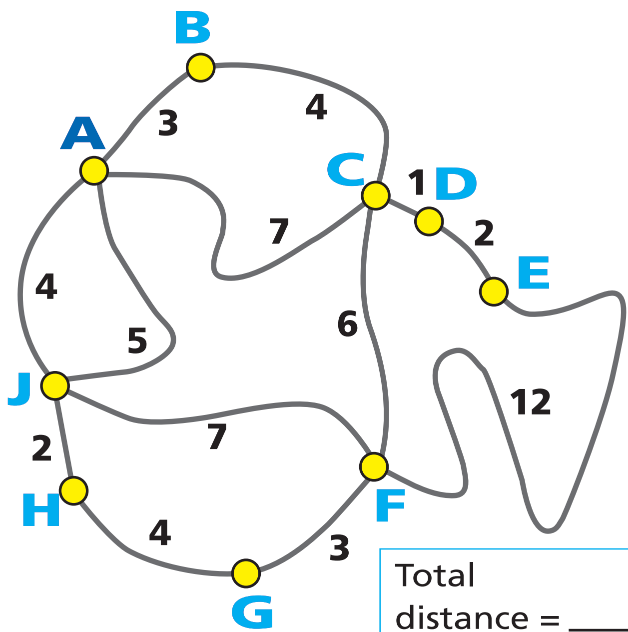
Start and end at **A**



Total distance = _____

4

Start and end at **A**



Total distance = _____

Teacher notes

Getting there : Working for efficiency

Description

This topic looks at simplified versions of three different network problems that are encountered in practical logistical planning.

Activity 1: Paper rounds

Activity 2: Cable connections

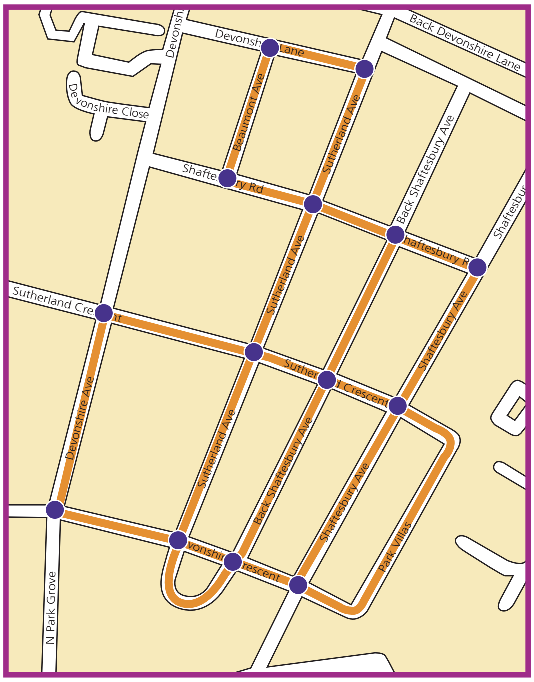
Activity 3: Deliveries

Edge-traceable graphs are explored in Paper rounds. The starter activity offers a simple version of a practical context in which to explore the concept. The famous mathematician Euler noticed that, to be edge-traceable, a network must have at most two odd nodes.

The worksheet offers opportunities for further exploration and then asks the pupils to search for a rule and to prove their results. At some point it is worth asking: *can you have just one odd node?*

Using tracing paper supports experimenting. For lower attaining pupils, slipping the worksheet into a plastic wallet will allow experimentation using felt tip pens, wiping off incorrect attempts, which may be easier.

Encourage pupils to look for what the edge-traceable networks have in common. Which graphs can be traced whatever starting point is chosen? Which graphs can be traced only if you start, and end, at particular points? Pupils will test out their theories better if they work together.



Cable connections provides a practical example of a minimum connector problem. It is quite possible to find a good or even an optimal solution using trial and improvement methods but the task provides the opportunity to introduce pupils to the key mathematical idea of an algorithm. Prim's algorithm, to find this minimum spanning tree, is quite within the grasp of many pupils, even at key stage 3.

For some of the problems on the worksheet, there is more than one optimum solution.

Prim's algorithm works as follows:

Step 1: *Choose an arbitrary location, say Grimsby, and connect it to the nearest place, in this case, Westgate.* ***Note the distance****.*

Step 2: *Find the nearest place not yet in the solution, in this case either Frenchy place or Wheelsby Road. Whenever you get a choice of connections of equal length, choose either one. Here we have arbitrarily chosen Frenchy place.*

Continue this process until all the locations are connected.

Getting there : Working for efficiency

Deliveries illustrates a very familiar problem in logistics. Finding the shortest route to a set of delivery points will save on both time and money. For a relatively small network like the problem offered here, the shortest route may be found by trial and improvement methods, or by considering all the possible routes. Pupils will develop helpful strategies, for example, they might decide to try to avoid longer stages, like the road which goes directly from Sheffield to Catcliffe.

You can extend this activity for pupils who solve the initial problem by making one or two of the roads one-way, or by making one of the roads impassable. Both of these variations do, of course, represent real features of shortest route problems as applied to road systems.

In this type of problem, the number of calculations needed to exhaust every possibility increases very fast. Because of the very large number of computations required to be sure of the best solution, mathematicians have developed algorithms which give good solutions for problems that are too large to test for every possible solution. Pupils may be intrigued to find out that, to date, there is no known algorithm which can guarantee a best solution for a large number of locations – an unsolved problem in mathematics. If they have derived some helpful strategies, can they find networks where their approach fails to provide the optimal solution?

The mathematics

Paper rounds offers the opportunity for reasoning and proof as the arguments needed to establish Euler's theorem are within their grasp. It also, along with Cable connections and Deliveries, offers opportunities for the mathematical skills of planning, being systematic, recording and logical experiment. The algorithmic thinking developed is picked up in key stage 5 in the decision maths curriculum.